Week 6: Slides about Chapter 7 of the Lecture notes.
**Complexity of finding cocliques**

**COCLIQUE**: Does a given $G = (V, E)$ have a coclique of size at least $k$? (Is $\alpha(G) \geq k$?)

Recall from last week:

**Theorem** **COCLIQUE** is $NP$-complete.

**Proof** Reduce 3-SAT to COCLIQUE.

Recall that $\tau(G)$ is the vertex cover number. By Gallai’s theorem

$$\alpha(G) = |V| - \tau(G).$$

**Corollary** Finding $\tau(G)$ is $NP$-complete.

**NB:** Recall that finding $\tau(G)$ is in $P$ for *bipartite graphs* (cf. Exercise 3.20).
Complexity of graph colouring

**GRAPH COLOURING:** Does a given $G = (V, E)$ allow a legal colouring using at most $k$ colours?

(Legal colouring = endpoints of a given edge have different colours and all vertices are coloured.)

**Theorem** GRAPH COLOURING is $NP$-complete.

**Definition** The chromatic (colouring) number $\gamma(G)$ of a graph $G = (V, E)$ is the minimal number of colours needed to legally colour $G$.

**Theorem** A planar graph can be 4-coloured (Appel & Haken, 1977).

**Theorem** It is $NP$-complete to decide if a given planar graph $G$ has $\gamma(G) = 3$. 
Applications of graph colouring

- Map colouring (Paris Metro map in Exercise 7.1)

- Storage of chemicals (Some chemicals may not be stored together)

- Radio frequency assignment. Transmitters that are close together must get different frequencies to avoid interference.

- Traffic light management.
At an intersection as indicated traffic flows are allowed from A to B, etc. as indicated. If the traffic is controlled by traffic lights none of these flows may intersect. The minimum number of ‘traffic waves’ is the coloring number of the graph at the right: nodes are the possible traffic flows, and arcs correspond to intersection flows.
Edge colouring

An edge colouring is a partition

$$\Pi = \{M_1, \ldots, M_p\}$$

of $E$, where each $M_i$ is a matching.

Edge colouring (chromatic) number:

$$\chi(G) := \min \{ |\Pi| : \Pi \text{ an edge colouring of } G \}$$

Let $\Delta$ denote the maximal degree of any node (valency).

$$\chi(G) \geq \Delta(G).$$

**Theorem** For bipartite graphs $\chi(G) = \Delta(G)$.

**Theorem** [Vizing, 1964] For simple graphs $\Delta(G) \leq \chi(G) \leq 1 + \Delta(G)$.
‘Dual’ edge colouring theorem

Definition: $\xi(G)$ as the maximal number of pairwise disjoint edge covers in $G$.

So, $\xi(G)$ is the maximum number of colours that can be used in colouring the edges of $G$ in such a way that at each vertex all colors occur.

Let $\delta(G)$ be the minimum degree of any node. Then obviously,

$$\xi(G) \leq \delta(G).$$

Theorem For bipartite graphs $\xi(G) = \delta(G)$ (König).

Application: Scheduling classes. (Exercise 7.9(i,ii,iii))
Partial orderings and chains

Definition: A partially ordered set is a pair \((X, \leq)\) such that the ‘\(\leq\)’ relation satisfies:

- \(x \leq x\) for each \(x \in X\) (reflexivity);
- if \(x \leq y\) and \(y \leq x\) then \(x = y\) (anti–symmetry);
- if \(x \leq y\) and \(y \leq z\) then \(x \leq z\) (transitivity).

Definition: \(C \subset X\) is called a \textit{chain} if for all \(x, y \in C\) either \(x \leq y\) or \(y \leq x\).

Definition: \(A \subset X\) is called an \textit{anti–chain} if for all \(x, y \in A\) with \(x \neq y\) neither \(x \leq y\) nor \(y \leq x\).

N.B.: \(|C \cap A| \leq 1|\).
Dilworth decomposition theorem

**Theorem:** Let \((X, \leq)\) be a partially ordered set with \(X\) finite. The minimum number of *anti–chains* needed to cover \(X\) equals the maximal cardinality of any *chain*.

We can interchange the words ‘*chain*’ and ‘*anti–chain*':

**Theorem (Dilworth):** Let \((X, \leq)\) be a partially ordered set with \(X\) finite. The minimum number of *chains* needed to cover \(X\) equals the maximal cardinality of any *anti–chain*.

**Applications:** Project scheduling, Hotel reservations, Terminal/platform assignment.
Perfect graphs

The clique number $\omega(G)$ of a graph $G$ is the maximal number of vertices in a clique. One has

$$\alpha(G) = \omega(G) \leq \gamma(G).$$

For which graphs do we have $\omega(G) = \gamma(G)$?

**Definition:** $G = (V, E)$ is called *perfect* if $\omega(G') = \gamma(G')$ for each induced subgraph $G'$ of $G$.

**N.B.:** For perfect graphs we can compute $\omega(G) = \gamma(G)$ in polynomial time (by using the $\vartheta$-function of Lovász).

**Theorem:** Let $H$ arise from $G$ by *duplicating a vertex*. Then $H$ is perfect if $G$ is.

**Theorem:** The complement of a perfect graph is also perfect.

**Theorem** (Chudnovsky, Robertson, Seymour, Thomas; 2003): $G$ is perfect if and only if $G$ does not contain an odd circuit of length $\geq 5$, or its complement, as an induced subgraph.
Examples of perfect graphs

Definition: A graph is called chordal if every cycle of length $\geq 4$ has a chord (shorter sub-cycle).

Theorem: Chordal graphs are perfect.

Theorem: Bipartite graphs are perfect.

Corollary (König edge cover theorem): The edge cover number of a bipartite graph (no isolated vertices) equals its coclique number.