Reliability Analysis of k-out-of-n Systems With Single Cold Standby Using Pearson Distributions

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***Abstract*—k-out-of-n systems with cold standby units are typically studied for unit lifetime distributions that allow analytical tractability. Often, however, these distributions differ significantly from reality. In this paper, we present an analytical approach to compute the mean failure time for k-out-of-n systems with a single cold standby unit for the wide class of lifetime distributions that can be captured by the Pearson distribution. The method requires the first four statistical moments of the unit's lifetime distribution to be given, and computes the mean failure time using the Pearson distribution as an intermediate vehicle during the numerical integration. Experimental results for various instances of the Weibull distribution show that the numerical accuracy of the approach is high, with less than 0.5 percent error across a large range of k-out-of-n systems.**

***Index Terms*—Cold standby, failure time analysis, order statistics, Pearson approximation, redundant systems.**

Acronyms

BDD	Binary decision diagram.
TMR	Triple modular redundant.
MC	Monte Carlo.

Notations

T	Failure time of composite system.
n	Number of units.
$f(x)$	Lifetime pdf of a unit.
$F(x)$	Lifetime cdf of a unit.
T_i	Lifetime of primary unit i.
$T_{i:n}$	i-th order statistic of n iid variates.

I. Introduction

SYSTEM reliability analysis often concerns analyzing k-out-of-n systems. A k-out-of-n system is operational when at least k out of the n units work, where for series systems $k = n$ while for parallel systems $k = 1$. Compared to active redundancy, cold standby redundancy can offer a better cost-reliability trade-off when switching times are sufficiently short, assuming the standby units are new (or repaired as good as new). Many approaches have been published for the failure time analysis of k-out-of-n systems with cold standby units, either in an application context that allows repair [1], [4], [18], [21], [22], or where repair is not possible [5], [17].

For tractability reasons, the unit lifetime is typically assumed to have a negative-exponential distribution. While a constant failure rate assumption enables a simple analysis and the use of Markov models [2], [9], [11], [15], the resulting failure time errors of the composite system can be significant when the components have other types of distributions, in particular, when n is large [14].

In many engineering cases, the unit's lifetime is determined by cumulative damage, or by the damage in the multiple components within the unit. This fact leads to more complex distributions (e.g., [3] where bell-shaped Weibull distributions are used to model fatigue life). As a simple example of the estimation errors that arise from assuming negative-exponential distributions, consider a 5-unit series system where the unit's lifetime is (equally) distributed according to a Weibull distribution with shape parameter $r = 5$ (bell-shaped). If one would model the units according to a negative-exponential distribution (with same mean), the estimation error for the series system's failure time would already exceed 100%, while growing with system size. In [10], the failure time distribution of k-out-of-n systems with $n - k$ cold standby units is described without requiring negative-exponential distributions. However, this approximate approach provides acceptable results for large n only.

In this paper, we present an alternative, analytical method to compute the mean failure time of k-out-of-n systems with cold standby for *arbitrary* unit lifetime distributions. The method is based on approximating the unit's pdf by a Pearson distribution, determined by the first four statistical moments of the unit's lifetime. The mean failure time of the k-out-of-n system is computed using numerical integration. This generic approach offers a design tool for k-out-of-n systems with cold standby for a wide range of lifetime distributions. The approach carries a number of restrictions.

1) First, the lifetime distribution must be unimodal, and amenable to sufficiently accurate modeling by the Pearson distribution.
2) Second, while the k-out-of-n systems can have arbitrary n and k, the analytical approach is typically limited to one cold standby only, as will be explained later on.
3) Finally, while requiring four moments of the input failure distribution, the method only provides the first moment of the resulting failure time distribution. However, the first moment, and its high accuracy, offers valuable design feedback.

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Our method is inspired by earlier work [14], where k-out-of-n systems were studied without cold standby. That approach is also based on the use of Pearson distributions as an intermediate, analytical vehicle, effectively providing a "4-moments-in-4-moments-out" calculus for kth-order statistics and additions. While the calculus has been shown to provide an excellent cost-performance trade-off [14], the subsystem failure times are required to be *s-independent*. While the failure time of very specific instances of k-out-of-n systems with cold standby (such as the simple systems in [1], [5], [18], [22]) can be solved using this calculus, the s-independence requirement prohibits its application to *arbitrary* k-out-of-n systems with cold standby redundancy, which motivates our paper. A typical example of such a system is shown in Fig. 1, comprising two primary units U_1, U_2, and a cold standby U_3. All units are identical, statistically and physically. The two units must be operational for the entire system to function (series system). When either U_1 or U_2 fails (U_1 in the example), U_3 replaces it, such that the system stays operational. When either U_2 or U_3 subsequently fail (U_2 in the example), the system fails (at time T in the figure). The system can be used as a low-cost alternative to triple modular redundant (TMR) systems in, e.g., aerospace [16] where payload weight and power consumption are critical. Rather than having all three units online, only two units are active. As long as both units provide consistent results, the system is operational. When an inconsistency is detected, the cold spare is activated, detecting the failing unit through majority voting. Compared to classical TMR, standby redundancy increases the operational lifetime without additional cost (of course, the time to switch on the cold standby unit must be acceptably short).

Let the lifetimes of U_1, U_2, and U_3 be T_1, T_2, and T_3, respectively. Failure of the series composition of the first two units U_1 and U_2 occurs at time

$$t_1 = \min(T_1, T_2),$$

after which the cold spare U_3 is switched on. Failure of the remaining unit U_2 occurs at

$$t_2 = \max(T_1, T_2).$$

Failure of U_3 occurs at time

$$t_1 + T_3$$

Consequently, the system fails at

$$T = \min(t_2, t_1 + T_3). \tag{1}$$

While the individual terms t_1, t_2, and the addition of t_1 to T_3 are covered by the moments calculus, the *correlation* between the t_1 and the $t_2 + T_3$ expressions prohibits the application of the calculus to the top-level "min" expression (a problem that would not exist, had negative-exponential lifetime distributions been assumed).

Extending the calculus to handle correlation is problematic. While, e.g., additions pose no analytical difficulty in computing the resulting moments when the correlation of the arguments is given, computing these correlations is difficult. Furthermore, little is known about order statistics of sequences of correlated variables. In [7], a Markov model is described where correlations are included in the model. However, this approach (again) is restricted to negative exponential distributions, in contrast to the aim of our paper.

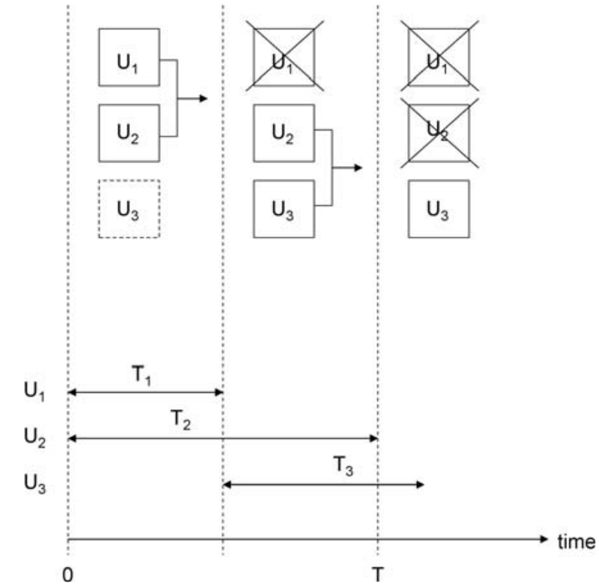

Fig. 1. Example series system with cold standby.

In this paper, we show how the above correlation problem can be solved when considering one cold spare only, enabling an analytic solution to the system's mean failure time $\mathsf{E}[T]$ for general distributions. We assess the solution's accuracy for a wide range of k-out-of-n systems with single cold standby with unit lifetime distributions taken from Weibull distributions that range from a negative-exponential shape to a bell-shape. Our results show that the estimation accuracy for $\mathsf{E}[T]$ is within half of a percent.

The paper is organized as follows. In Section II, we briefly summarize the Pearson distribution and some main results on order statistics. In Section III, we present our solution to computing $\mathsf{E}[T]$, while in Section IV we empirically evaluate its accuracy. In Section V, we summarize our findings.

II. Preliminaries

In this section, we briefly describe the Pearson distribution, and summarize some order statistics results that we use in the paper.

A. Pearson Distribution

With its four parameters, the Pearson method can model a wide range of distribution shapes. It can model non-standard lifetime distributions of real components, which motivates our choice for this distribution as an intermediate vehicle in the analytic derivation of $\mathsf{E}[T]$.

It has been shown by Pearson that the derivative of any density function $f(x)$ can be approximated by [19]

$$\frac{df(x)}{dx} = \frac{(x+a)f(x)}{d_0 + d_1 x + d_2 x^2} \tag{2}$$

The parameters a, d_0, d_1, and d_2 can be derived from the first four raw moments m_1, m_2, m_3, and m_4 of the target density function. To greatly improve numerical stability, one rather derives the above Pearson parameters for a version of the target function that has been shifted by m_1 such that its first moment becomes zero. This amounts to computing the Pearson parameters from the four target moments given by

$$\begin{aligned} u_1 &= 0 \\ u_2 &= m_2 - m_1^2 \\ u_3 &= m_3 - 3m_1m_2 + 2m_1^3 \\ u_4 &= m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4 \end{aligned} \tag{3}$$

Given u_1, u_2, u_3, and u_4, the Pearson parameters are derived according to [13]

$$\begin{aligned} a &= \left(3u_2^2u_3 + u_3u_4\right)/q \\ d_0 &= \left(-4u_2^2u_4 + 3u_2u_3^2\right)/q \\ d_1 &= \left(-3u_2^2u_3 - u_3u_4\right)/q \\ d_2 &= \left(3u_3^2 + 6u_2^3 - 2u_2u_4\right)/q \end{aligned} \tag{4}$$

where $q = -18u_2^3 + 10u_2u_4 - 12u_3^2$. Depending on whether

$$d_0 + d_1x + d_2x^2 = 0 \tag{5}$$

has real roots with different signs, complex roots, or real roots with the same sign, the integration of (2) results in different analytical expressions [19]. The expression to be obtained depends on the criterion

$$\eta = \frac{d_1^2}{4d_0d_2}.$$

By defining

$$\text{skew} = \sqrt{B_1} = \frac{u_3}{u_2^{3/2}}, \tag{6}$$

$$\text{kurtosis} = B_2 = \frac{u_4}{u_2^2}, \tag{7}$$

η becomes

$$\eta = \frac{B_1(B_2+3)^2}{4(2B_2 - 3B_1 - 6)(4B_2 - 3B_1)}. \tag{8}$$

In the following, we describe the different distribution types, as modified by us to improve numerical stability. Details can be found in [13].

If $\eta < 0$, $f(x)$ becomes a Pearson type I distribution according to

$$f(x) = c_1 \left[\frac{2(a_1+x)}{a_1+a_2}\right]^A \left[\frac{2(a_2-x)}{a_1+a_2}\right]^B, \tag{9}$$

for the domain $-a_1 < x < a_2$ where

$$\begin{aligned} a_1 &= \frac{d_1 + \sqrt{d_1^2 - 4d_0d_2}}{2d_2} \\ a_2 &= \frac{-d_1 + \sqrt{d_1^2 - 4d_0d_2}}{2d_2} \\ A &= \frac{a_1 - a}{d_2(a_1+a_2)} \\ B &= \frac{a_2 + a}{d_2(a_1+a_2)}, \end{aligned} \tag{10}$$

and where c_1 is (numerically) computed such that $f(x)$ integrates to 1 within its domain.

If $0 < \eta < 1$, $f(x)$ becomes a Pearson type II distribution according to

$$f(x) = c_1 \left[1 + \left(\frac{x}{c} - \frac{v}{r}\right)^2\right]^{-m} e^{-v \arctan(x/c - v/r)} \tag{11}$$

where

$$\begin{aligned} m &= -\frac{1}{2d_2} \\ r &= 2m - 2 \\ c &= \frac{1}{4}\sqrt{u_2(16(r-1) - B_1(r-2)^2} \\ v &= \pm\, r(r-2)\frac{\sqrt{B_1u_2}}{4c} \end{aligned} \tag{12}$$

with v having the opposite sign of the skew. We have limited the domain of $f(x)$ to $-15\sqrt{u_2} < x < 15\sqrt{u_2}$.

If $\eta > 1$, $f(x)$ becomes a Pearson type III distribution according to

$$f(x) = c_1 \left(\frac{a_1+x}{a_1-a_2}\right)^A \left(\frac{2(a_2+x)}{a_2+15\sqrt{u_2}}\right)^B, \quad -a_2 < x < 15\sqrt{u_2} \tag{13}$$

if the skew is positive, and

$$f(x) = c_1 \left(\frac{2(a_1+x)}{-15\sqrt{u_2}+a_1}\right)^A \cdot \left(\frac{a_2+x}{a_2-a_1}\right)^B, \quad -15\sqrt{u_2} < x < -a_1 \tag{14}$$

if the skew is negative.

If $\eta = 0$, depending on the value of B_2, this case is treated as Pearson type I or type II. All three types have occurred during our experiments. More implementation details can be found in [13].

B. Order Statistics Results

From the example in the Introduction, it is clear that analysis of $\mathbf{E}[T]$ generally involves computing the distributions of the i-th order statistic $T_{i:n}$ of a series of n s-independent, identically distributed (iid) variables, as well as of the minimum of two variables ("min").

The failure time distribution of k-out-of-n systems equals $T_{n-k+1:n}$, i.e., the $(n-k+1)$-th order statistic of the n iid lifetimes.

The pdf of the $(n-k+1)$-th order statistic $T_{n-k+1:n}$ of n iid variables is given by [6]

$$\frac{n!}{(k-1)!(n-k)!}F^{n-k}(x)\left[1-F(x)\right]^{k-1}f(x) \tag{15}$$

Furthermore, the pdf of the minimum $\min(T_1, T_2)$ of two s-independent, but not necessarily identical variables, T_1 and T_2 is given by [14]

$$\left[1-F_1(x)\right]f_2(x) + \left[1-F_2(x)\right]f_1(x). \tag{16}$$

For identical units, this equation reduces to (15) for $n = 2$, and $k = 2$ (a series system).

III. Failure Time Analysis

Fig. 2 generalizes the 2-out-of-2 system, which was given in the Introduction (see Fig. 1). Fig. 2 shows a k-out-of-n system with 1 cold standby. All units have equal lifetime distributions. Again, without loss of generality, let the n units U_i be ordered in increasing lifetime sample T_i, as shown in the figure. Consequently, T_i equals the i-th order statistic $T_{i:n}$.

At T_{n-k+1}, the first k units have failed, after which the cold spare is turned on. The failure time of the system is given by

$$T = \min(T_{n-k+2},\ T_{n-k+1} + T_{n+1}) \quad (17)$$

where T_{n+1} denotes the lifetime of the cold unit. As mentioned earlier, the above expression of T suffers from the fact that the arguments of the "min" expression are correlated. To solve this problem, we rewrite the expression for T as follows. Because, unlike a "min" operator, a "+" operator allows correlating terms to be handled, we rewrite the expression to

$$T = T_{n-k+1} + \min(T_{n-k+2} - T_{n-k+1},\ T_{n+1}), \quad k = 1, 2, \ldots, n-1 \quad (18)$$

by subtraction of T_{n-k+1}. Equation (18) has two advantages.

1 The correlation within the "min" expression is removed. Consequently, the existing moments calculus in [14] can be applied to the "min" expression (16).

2 Despite their mutual correlation, the difference between T_{n-k+2} and T_{n-k+1} can be computed because the pdf of the difference $z = T_{i+1:n} - T_{i:n}$ is given by [6]

$$\frac{n!}{(i-1)!(n-i-1)!} \int_{-\infty}^{\infty} F(x)^{i-1} f(x) f(x+z) \times (1 - F(x+z))^{n-i-1} dx. \quad (19)$$

We are still left with the fact that both arguments of the addition in (18) are still correlated, the extent of which is hard to determine. Consequently, only the first moment $\mathsf{E}[T]$ of T can be solved.

In summary, computing $\mathsf{E}[T]$ involves the following steps.

1 Compute the raw moments of $T_{n-k+2} - T_{n-k+1}$ using (19), where $f(x)$ is the Pearson distribution derived from the first four moments of the unit's lifetime (cf. Section II-A).

2 Compute the first moment of $\min(T_{n-k+2} - T_{n-k+1}, T_{n+1})$ using (16), where $f_1(x)$, and $f_2(x)$ are the Pearson distributions fitting the previously computed moments of $T_{n-k+2} - T_{n-k+1}$ in Step 1, and the moments of the (cold) unit lifetime, respectively.

3 Compute the first moment of T_{n-k+1} using (15), where $f(x)$ is the Pearson distribution derived from the first four moments of the unit's lifetime. Add this moment to the first moment computed in Step 2 resulting in $\mathsf{E}[T]$.

While the reliability analysis for k-out-of-n systems with one cold standby can be handled by the above approach, the situation for multiple standby units becomes more complicated. Fig. 3 depicts the timing of a 3-unit series system with two cold standby units U_4, and U_5, with lifetimes T_4, and T_5, respectively. The failure time when U_4 has been switched on is given by $T_1 + \min(T_2 - T_1, T_4)$. At this time, U_5 is switched on. However, the combinational "explosion" caused by the various failure sequences of U_2, U_3, U_4, U_5 prohibits a simple, unconditional expression in terms of just order statistics and additions, as shown in (18). Fig. 4 shows the binary decision diagram (BDD)-like expression tree for T. The depth of the tree is proportional to the minimum of n and the number of cold standby units m, as system failure occurs when less than n primary units and standby units are operational. At first glance, the expression tree would seem amenable to the moments calculus. Each condition can be rewritten in terms of $T_{k+1} - T_k < T_i$. As mentioned earlier, the moments of the left-hand side of the comparison can be computed. Because the left-hand and right-hand side terms are s-independent, the moments of the condition can be evaluated using Pearson distributions, similar to the functions described earlier. Furthermore, the moments of a conditional expression could be expressed in terms of the moments of the condition, the "then" clause, and the "else" clause, if the conditions and clauses were s-independent [8]. However, the recursive application of the moments computation for each condition poses the same dependence problem as found earlier. Because the leaves and the conditions downstream in the tree inherit terms from their predecessor conditions, the correlation between the conditions and clauses prohibits a straightforward computation of the moments. Also, when the tree would be collapsed to an exponential list of individual (leaf) expressions according to

$$T = \begin{cases} T_1 + T_4 + T_5 & T_2 - T_1 > T_4 \wedge T_2 - T_1 > T_4 + T_5, \\ T_2 & T_2 - T_1 > T_4 \wedge T_2 - T_1 < T_4 + T_5, \\ T_2 + T_5 & T_2 - T_1 < T_4 \wedge T_3 - T_2 > T_5, \\ T_3 & \text{otherwise}; \end{cases} \quad (20)$$

for most clauses the problem would translate to evaluating the moments of a logical expression with correlating terms, which, again, prohibits computing the moments. In principle, this problem could be addressed by deriving the joint probability distribution of the three consecutive i-th, $(i+1)$-th, and $(i+2)$-th order statistics (an extension of the derivation for i and $i+1$ in (19)). But whether the computational complexity of a solution for large systems still outweighs numerical simulation becomes questionable.

As becomes clear from the above, there are two cases where the above tree reduces to an unconditional expression, i.e., when either n or m are one. Thus the only multiple cold standby configuration that can be handled is for a single primary unit $(n = 1)$, for which it can be easily seen that the failure time is given by

$$T = \sum_{i=1}^{n+m} T_i \quad (21)$$

where m denotes the number of standby units, which makes the computation of $\mathsf{E}[T]$ trivial. The other case $(m = 1)$ is the focus of this paper.

A. Implementation

Fig. 5 lists the algorithm that implements the computation of $\mathsf{E}[T]$ (15), (16), (19). The first four raw moments of the unit lifetime distributions are denoted by m_1, m_2, m_3, m_4. Given the four moments, PEARSON() computes the Pearson distribution $f(x)$, and the associated lower and upper bound of its domain, as explained in Section II-A.

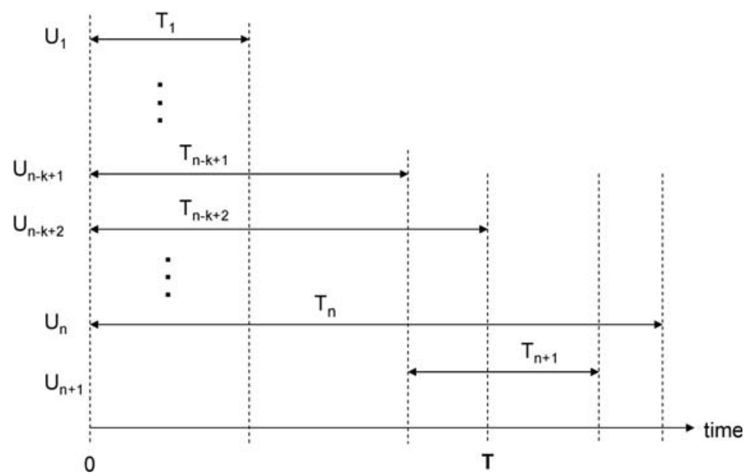

Fig. 2. k-out-of-n system with one cold standby unit.

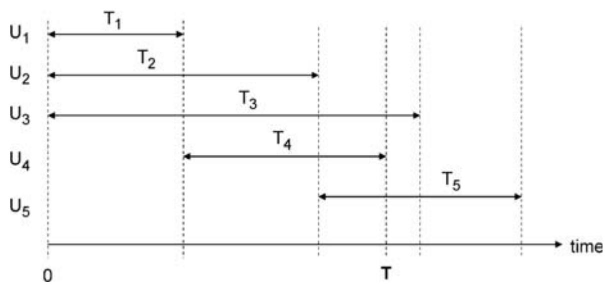

Fig. 3. Example series system with two cold standby units.

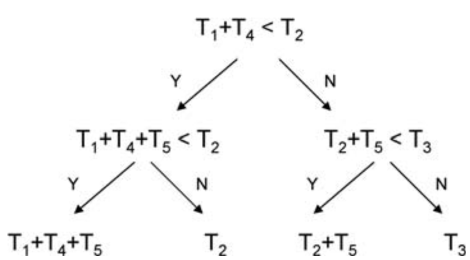

Fig. 4. Expression tree for example n-out-of-n system timing, two cold standby units.

The first two integration loops (lines 5–16) correspond to (19), and compute the moments n_1, n_2, n_3, n_4 of $T_{n-k+2} - T_{n-k+1}$.

The second integration loop (lines 23–28) computes $\min(T_{n-k+2} - T_{n-k+1}, T_{n+1})$ according to (16). Because both arguments to the "min" operation have different mean values, the Pearson distribution of the second argument (T_{n+1}) is shifted by $s = n_1 - m_1$ to ensure that f_1 and f_2 are s apart in the numerical integration. In the computation of the first moment (p_1, line 27), n_1 is added back to x to compensate for the fact that the Pearson distribution has been centralized around $x = 0$. Details are found in [14].

The third integration loop (lines 32–36) computes the order statistic T_{n-k+1}. Again, the original first moment is added to x in the computation of the first moment (q_1, line 35). Finally, $\mathbf{E}[T]$ is derived in line 37.

Our experiments show that $N = 10\,000$ iterations are sufficient to guarantee that numerical accuracy is not limited by integration errors. Note that our implementation is based on simple Euler integration as our primary aim is to demonstrate the inherent accuracy of our approach, rather than optimizing the $cost/performance$ ratio.

IV. Experimental Results

Tables I–III show the accuracy of $\mathbf{E}[T]$ for the interesting range of k-out-of-n systems $(k = 2, \ldots, n)$, for Weibull distributions given by

$$f(x) = r\lambda(\lambda x)^{r-1}e^{(-\lambda x)^r} \tag{22}$$

with scale parameter $\lambda = 1$, and shape parameter ranging from $r = 1$ (negative-exponential) to $r = 5$ (bell-shape). The tables list the Weibull shape parameter r, system parameters n and k, $\mathbf{E}[T]$ as measured by Monte Carlo (MC) simulation of (18) (denoted $\mathbf{E}[T]_{\mathrm{M}}$), $\mathbf{E}[T]$ as computed by our algorithm (denoted $\mathbf{E}[T]_{\mathrm{A}}$), and the corresponding estimation error ϵ in percents.

$\mathbf{E}[T]_{\mathrm{A}}$ is averaged over 10 000 000 samples, yielding a 95% confidence interval of less than 0.1%.

MC simulation uses samples taken from the Weibull distribution. The algorithm's input moments m_1, m_2, m_3, m_4 are based on the Weibull distribution's raw moments

$$m_i = \left(\frac{1}{\lambda}\right)^i \Gamma(1 + i/r), \quad i = 1, \ldots, 4 \tag{23}$$

The estimation error is given by

$$\epsilon = \left| \frac{\mathbf{E}[T]_{\mathrm{A}} - \mathbf{E}[T]_{\mathrm{M}}}{\mathbf{E}[T]_{\mathrm{M}}} \right| \cdot 100\% \tag{24}$$

As can be seen from all tables, the approach exhibits excellent accuracy, in full agreement with the accuracy of the Pearson approach as reported in [14] for Gaussian distributions. The error is in the same range as the error obtained when directly comparing the moments as measured from the Pearson distribution with the Pearson distribution's input moments based on the Weibull distribution (23).

```
f, x_l, x_u ← PEARSON(m_1, m_2, m_3, m_4)
dx ← (x_u − x_l)/N
dz ← 2 · x_u/N
n_1, ..., n_4 ← 0
for z ← 0 to 2 · x_u step dz do
        g, F_x, F'_x ← 0
        for x ← x_l to x_u step dx do
                F_x ← F_x + f(x) · dx
                F'_x ← F'_x + f(x + z) · dx
                g ← g + n! · F_x^{k−1} · f(x) · f(x + z) · (1 − F'_x)^{n−k−1} · dx/((k − 1)! · (n − k − 1)!)
        od
        n_1 ← n_1 + z · g · dz
        n_2 ← n_2 + z^2 · g · dz
        n_3 ← n_3 + z^3 · g · dz
        n_4 ← n_4 + z^4 · g · dz
od
f_1, x_{1,l}, x_{1,u} ← PEARSON(n_1, n_2, n_3, n_4)
f_2, x_{2,l}, x_{2,u} ← PEARSON(m_1, m_2, m_3, m_4)
s ← n_1 − m_1
x_l ← min(x_{1,l}, x_{2,l} + s);   x_u ← max(x_{1,u}, x_{2,u} + s)
dx ← (x_u − x_l)/N
F_1, F_2, p_1, ..., p_4 ← 0
for x ← x_l to x_u step dx do
        F_1 ← F_1 + f_1(x) · dx
        F_2 ← F_2 + f_2(x − s) · dx
        g ← (1 − F_1) · f_2(x − s) + (1 − F_2) · f_1(x)
        p_1 ← p_1 + (x + n_1) · g · dx
od
f, x_l, x_u ← PEARSON(m_1, m_2, m_3, m_4)
dx ← (x_u − x_l)/N
F, q_1, ..., q_4 ← 0
for x ← x_l to x_u step dx do
        F ← F + f(x) · dx
        g ← n! · F^{k−1} · (1 − F)^{n−k} · f(x)/((n − k)! · (k − 1)!)
        q_1 ← q_1 + (x + m_1) · g · dx
od
E[T] ← q_1 + p_1
```

Fig. 5. Algorithm for the computation of $\mathsf{E}[T]$.

TABLE I
ACCURACY [%] FOR THE WEIBULL DISTRIBUTION $r = 1$

r	n	k	$\mathsf{E}[T]_{\mathrm{M}}$	$\mathsf{E}[T]_{\mathrm{A}}$	ϵ
1	2	2	1.000	0.998	0.2
1	3	2	1.333	0.331	0.2
1	3	3	0.667	0.665	0.3
1	5	2	1.783	1.782	0.1
1	5	3	1.116	1.116	0.0
1	5	4	0.700	0.698	0.3
1	5	5	0.400	0.399	0.3
1	10	2	2.429	2.425	0.2
1	10	3	1.763	1.758	0.3
1	10	4	1.345	1.346	0.1
1	10	5	1.046	1.045	0.1
1	10	6	0.812	0.812	0.0
1	10	7	0.622	0.621	0.2
1	10	8	0.461	0.461	0.0
1	10	9	0.322	0.322	0.0
1	10	10	0.200	0.201	0.4

TABLE II
ACCURACY [%] FOR THE WEIBULL DISTRIBUTION $r = 2$

r	n	k	$\mathsf{E}[T]_{\mathrm{M}}$	$\mathsf{E}[T]_{\mathrm{A}}$	ϵ
2	2	2	1.044	1.044	0.0
2	3	2	1.220	1.220	0.0
2	3	3	0.819	0.818	0.1
2	5	2	1.413	1.413	0.0
2	5	3	1.077	1.075	0.2
2	5	4	0.835	0.834	0.1
2	5	5	0.616	0.616	0.0
2	10	2	1.642	1.644	0.2
2	10	3	1.355	1.355	0.0
2	10	4	1.169	1.167	0.2
2	10	5	1.023	1.020	0.3
2	10	6	0.897	0.894	0.3
2	10	7	0.781	0.780	0.1
2	10	8	0.668	0.669	0.1
2	10	9	0.553	0.554	0.2
2	10	10	0.428	0.429	0.1

V. CONCLUSION

The analysis of k-out-of-n systems with cold standby units typically involves negative-exponential unit lifetime distributions. This approach may lead to severe prediction errors where the actual distribution shapes are different (wear and tear, cumulative damage, units that are complex systems themselves). In this paper, we have presented an analytical solution to the mean failure time $\mathsf{E}[T]$ of k-out-of-n systems with a single cold standby for arbitrary distributions. The method assumes the first four statistical moments of the unit's lifetime distributions to be known, and has been implemented for the wide range of unit lifetime distributions that can be modeled by the Pearson distribution. MC simulation experiments for a wide range of k-out-of-n systems with 1 cold standby have shown that the estimation error of $\mathsf{E}[T]$ is consistently less than 0.5% for the various distribution shapes taken from the Weibull distribution, ranging from negative-exponential to bell shape. For multiple

TABLE III
ACCURACY [%] FOR THE WEIBULL DISTRIBUTION $r = 5$

r	n	k	$\mathsf{E}[T]_\mathrm{M}$	$\mathsf{E}[T]_\mathrm{A}$	ϵ
5	2	2	1.036	1.035	0.1
5	3	2	1.093	1.094	0.1
5	3	3	0.923	0.922	0.1
5	5	2	1.154	1.156	0.2
5	5	3	1.028	1.028	0.0
5	5	4	0.926	0.924	0.2
5	5	5	0.817	0.815	0.2
5	10	2	1.223	1.225	0.2
5	10	3	1.127	1.129	0.2
5	10	4	1.062	1.062	0.0
5	10	5	1.006	1.004	0.2
5	10	6	0.953	0.952	0.1
5	10	7	0.901	0.900	0.1
5	10	8	0.845	0.844	0.1
5	10	9	0.782	0.781	0.1
5	10	10	0.703	0.702	0.1

cold standby units, the method suffers from a combinatorial state explosion involving correlated terms that prohibit the approach to offer a practical solution, except for systems with a single primary unit.

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REFERENCES

[1] S. C. Agarwal, M. Sahani, and S. Bansal, “Reliability characteristic of cold-standby redundant system,” *IJRRAS*, vol. 3, no. 2, pp. 193–199, May 2010.

[2] M. Asadi and I. Bayramoglu, “The mean residual life function of a k-out-of- n structure at the system level,” *IEEE Trans. Reliability*, vol. 55, no. 2, pp. 314–318, June 2006.

[3] E. A. Azrulhisham, Y. M. Asri, A. W. Dzuraidah, and A. H. Hairul Fahmi, “Application of Pearson parametric distribution model in fatigue life reliability evaluation,” *World Academy of Science, Engineering and Technology*, vol. 71, pp. 243–248, 2010.

[4] W. K. Chung, “A k-out-of-N:G redundant system with cold standby units and common-cause failures,” *Microelectron. Reliab.*, vol. 24, no. 4, pp. 691–695, 1984.

[5] D. W. Coit, “Cold-standby redundancy optimization for non-repairable systems,” *IIE Trans.*, vol. 33, no. 6, pp. 471–478, 2001.

[6] H. A. David and H. N. Nagaraja, *Order Statistics*, 3rd ed. : John Wiley and Sons, 2003.

[7] J. Dugan, “Correlated hardware failures in redundant systems,” in *Second Int’l Working Conf. on Dependable Comput. for Critical Appl.*, Tucson, AZ, Feb. 1991, pp. 73–81.

[8] H. Gautama, “A Statistical Approach to the Performance Modeling of Parallel Systems,” Ph.D. thesis, Delft University of Technology, , The Netherlands, 2004.

[9] S. Eryilmaz, “Dynamic behavior of k-out-of- n : G systems,” *Operations Research Letters*, vol. 39, pp. 155–159.

[10] D. P. Kroese and W. C. M. Kallenberg, “Approximations to the lifetime distribution of K-out-of- N systems with cold standby,” *Mathematics of Operations Research*, vol. 14, no. 3, pp. 485–501, Aug. 1989.

[11] X. Li and M. J. Zuo, “On the behavior of some new ageing properties based upon the residual life of k-out-of- n systems,” *J. Applied Probability*, vol. 39, pp. 426–433, 2002.

[12] W-M. Lin, “Performance modeling and analysis of correlated parallel computations,” *Parallel Computing*, vol. 34, no. 9, pp. 521–538, September 2008.

[13] G. L. Reijns and A. J. C. van Gemund, “Predicting the execution times of parallel-independent programs using Pearson distributions,” *Parallel Computing*, vol. 31, no. 9, pp. 877–899, Sept. 2005.

[14] G. L. Reijns and A. J. C. van Gemund, “Reliability analysis of hierarchical systems using statistical moments,” *IEEE Trans. Reliability*, vol. 56, no. 3, September 2007.

[15] A. Sarhan, “Reliability equivalence factors of a parallel system,” *Reliability Engineering and System Safety*, vol. 87, pp. 405–411, 2005.

[16] G. Sinaki, “Ultra-reliable fault tolerant inertial reference unit for spacecraft,” in *Proc. Annual Rocky Mountain Guidance and Control Conf.*, San Diego, CA, 1994, pp. 239–248.

[17] Y. Shi, K. Kou, and X. Li, “Estimation of the reliability indexes for a cold standby system under type II censoring data,” *Journal of Physical Sciences*, vol. 10, pp. 85–92, 2006.

[18] V. Sridharan, “Availability analysis of series systems with cold standby components and general repair time,” *Applied Sciences*, vol. 9, pp. 155–162, 2007.

[19] A. Stuart and J. K. Ord, *Kendall’s Advanced Theory of Statistics*, 6th ed. New York: Halsted Press, 1994.

[20] K. S. Trivedi, *Probability and Statistics With Reliability, Queuing and Computer Science Applications*. : Prentice Hall, 1982.

[21] W. Wang and J. Loman, “Reliability/availability of k-out-of- n system with m cold standby units,” in *Proc. Annual Reliability and Maintainability Symp.*, 2002, pp. 450–455.

[22] Y. L. Zhang, “An optimal geometric process model for a cold standby repairable system,” *Reliability Engineering and Systems Safety*, vol. 63, pp. 107–110, 1999.

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